The 17-page translation, from German, which follows, is of the paper by Paul Gerber 1902, as republished in Annalen der Physik by its editor, Ernst Gehrcke, as:-

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For the purpose of this translation, I obtained free-of-charge bmp images of the journal pages retained in the archive of Biblithèque nationale de France. These were then saved in jpeg format to reduce somewhat the very big overall file size.
I then commissioned a German student, studying in England, to do the translation, be done into Word and supplied in pdf format only. To eliminate equation-copying errors and reduce cost these were imported as 'pictures' and pasted into place.
Textual changes from literal translation were made in only the interests of clarity and good English.
The student wishes his identity to remain private, without accreditation for the translation.

Miles Osmaston
miles@osmaston.demon.co.uk

The propagation velocity of gravitation*

P. Gerber

I.

In the "Zeitschrift für Mathematik und Physik", vol. 43, p. 93-104, I have shown: If one assumes that the hitherto unexplained movement of the perihelion of Mercury's orbit by 41" in a century is caused by the time-delay of the spatial propagation of gravitation, then it follows that the latter has the same velocity as light, thermal radiation and electric waves. One has to pay attention to what can really be proven through calculation and observation, and what has to be assumed without proof. If the gravitational force between two masses is constituted by an interaction that acts with a time-delay from one to the other, then one finds that a movement of the perihelion of a planet has to be the result. However, one cannot prove that the amount of the movement of Mercury's perihelion, which is not to be derived from a perturbation, cannot have another cause as the time-delay. If the assumption of this cause would lead to a different propagation velocity, then this would be of no interest. Only the agreement of both velocities justifies the assumption and thus the idea of a finite propagation velocity of gravitation.

This agreement is not only what is new, but also what is unexpected in my derivation. Since, however much one thought that the velocity of gravitation would reveal itself like the velocity of light, all previous exact attempts to determine it have led to different, usually higher values; such that it appeared as if there was no time-delay in the constitution of the attracting forces between the masses. Already Mach, in the fourth edition of his famous work on mechanics and its evolution, has highlighted the conflict between mine and previous works. Why is it that suddenly now this result is reached, which has been thought to be improbable for a long time? In my thesis, I have given the answer to this question only very briefly, in order to avoid far-reaching methodological and other explanations. Nevertheless, one will agree with me that I deem a detailed account to be necessary after this page.

Above all, the following has to be kept in mind. After various evaluations of the propagation velocity of gravitation have given such different results, varying between three fifth and ten million times the velocity of light, one may suppose

*This work of Gerber has been published 1902 as a programatic thesis of the "städtische Realgymnasium zu Stargard i. Pomm." after a shorter version was published in 1898 in the "Zeitschrift für Mathematik und Physik", vol. 43, p. 93-104. In reprinting this hard-to-get programmatic thesis in the "Annalen", I am following requests that have been expressed from various sides on the occasion of my paper in the "Annalen", vol. 51, p. 119-124, 1916. – Gerber draws a connection between the velocity of light and the velocity of gravitation, and is able to explain the perihelion movement of Mercury quantitatively. Whether and how Gerber's theory can be merged with the well known electromagnetic equations into one coherent theory is a difficult question, which yet has to be answered. E. Gehrcke [editor of Annalen] that not the calculations or observations have been false, but the fundamental assumptions. This will be confirmed in what follows. In any case, it is rash to jump to the conclusion that these results have astronomic certainty, as it is sometimes done. After my analysis has shown that certain ideas about the time-delay of the propagation of gravitation lead to the velocity of light, it is not so much relevant whether the velocity of gravitation reveals itself on planetary movements, the rotation of the moon or other cosmic events, but only on how far ideas can be developed and safely established about what propagates in the space between the masses, which quantities this has an immediate influence on, and of which quality and quantity this influence is. It will turn out that the old methods are not sufficient to the requirements, which are conditioned on the latter. I will not repeat the main points of my previous thesis here. They are mainly critical and historical explanations of fundamental issues, which deliberately have been presented only briefly or not at all.

II.

First of all, three main points have to be mentioned. The first concerns the concept of space, in which gravitational processes take place, at rest or movement of the masses; the second concerns the importance of the gravitational potential for the present problem; the third concerns the time-consumption in the communication of the forces to the masses, which is related to, but not the same as the duration of the propagation in space.

Usually one approaches the riddle of gravitation as if the main issue was to find a mechanical cause, which produces the gravitating of the masses. Many times the idea of pushing or wave impulses is in the background or is explicitly assumed when one talks about the propagation of gravitation. However, neither is it necessary nor possible to analyze how far gravitational processes really have the exceptional position, which they seemingly have above all other physical processes, or in how far they only appear to have this position. This means, it is important to understand gravitation in relation to the complete physical system. The proof of its successive propagation is the first step in order to end its exceptional status. Because it is a substantial difference whether it is looked at from the one or the other point of view. If one pictures two masses that are put into space suddenly out of nothing and do not attract each other in this moment, then this is all that is happening and comes from the simultaneous presence of both masses. On the other hand, if one envisions that the attraction only begins some time after the masses have been brought into space, then from each one a state has to propagate into its vicinity, which, contrary to the previous configuration of space, can be called a constrained state, and which as soon as it has reached the other mass shows itself in a movement thereof. In the first case, the space around one mass, if it is alone, remains the same, since this is the way it also is when another mass is present and it only realizes the distance between the masses; in the second case, the constrained state around one mass even happens when no other masses are present.

One should consider that the constrained state is not an empirical fact, but only follows from the concept of successive propagation and can only be established as a fact through the latter. Consequently, to assume propagation without the latter state and its consequences results in contradictions. In the same way as one refers to the space around electrically charged matter as an electric field, expressing that the charge has caused a change in it, one has to say the following: When a mass is brought into space, a gravitational field is generated, viz., a certain modification that spreads out beginning from the mass. What constitutes the latter is entirely beyond our knowledge; we perceive its presence only by the attraction that affects another mass in the gravitational field. Moreover, its concept is thereby fully defined; and one would misunderstand it, if one would regard and investigate it as something separated from the masses. Only spatial and temporal relations could be considered, which, however, would not teach anything new about their essence, namely the attraction of the masses. Such a relation is the temporal measure of the spatial propagation of the constrained state, on which the velocity of gravitation rests.

If this is too abstract, then one should remember that also about, e.g., the propagation of light nothing more can be said. Nevertheless, one can show that longitudinally to a light beam a periodical process has to take place; but one does not know what it is that grows and falls off in turns, even if one calls it electric or magnetic force, which also only refers to an undefinable state, not in need for a definition, despite the visible or conceivable effects on a mass that has to be brought to the specific place. But the abstraction, that the spatial propagation of the attraction of masses is constituted by a certain unspecified constrained state of space, is not empty enough to not reveal the effect of the velocity of gravitation on the movement of the masses. The limitation to the pure concept only prevents that more is deduced then what is appropriate to the matter and the assumptions.

Now, let two masses, which, to simplify things, can be constructed as infinitely small and one is called the attracting and the other the attracted mass, in a certain distance from each other, be at rest. Then the constrained state has some time to propagate from one mass to the other and conversely. One can assume it to be in the different spatial elements around the masses, and also longitudinally to the distance, taken with a suitable measure. If one separated the distance into infinitely thin zones, then the measure of the state within each zone can be regarded as constant, and, for example, at the attracted mass zone s, stemming from the attracting mass, and zone 1, stemming from the attracted mass, come together to form a combined constrained state. The movement the masses make, after they have been released, follows Newton's law, which is empirically known. Hence, if the attracting mass, for the width of t zones closer to the other, and the attracted mass, for the width of 1 zone closer to the other, are held at rest for a long time and then released, at the attracting mass the zone $s - t - 1$, stemming from the attracting mass, and the zone 1, stemming from the attracted mass, come together and again the movement for the new distance follows Newton's law. After the movement has started, in both cases the constrained states begin to propagate in space, coming from the masses, in constantly renewing distributions. If, however, the masses move freely from being at rest in the first position into the second position, the attracted mass does not interact with the zone $s - t - 1$, stemming from the attracting mass, but with the zone $s-1$; because the new configurations of the constrained state, coming from the attracting mass, have not yet reached the attracted mass and therefore the latter is still in the initial constrained state. Nevertheless – and this is of great importance – the constrained state $s - 1$, coming from the attracting mass, does not meet the state 1, coming from the attracted mass, by which the constrained state in the vicinity of the attracted mass would configure such as if the distance of the masses at rest was shortened by the width of one zone compared to the original distance. Since, while the attracted mass moves through the width of one zone, constantly new configurations of the constrained state come from it, such that, when the mass has reached the end of the zone, in the width of the succeeding zone the state of an initial zone only begins to develop, and of the states, coming from preceding positions of the mass, one part of the now conterminal zone is filled by the state of an initial zone another part, however, by the state of a second zone.

The latter can also be realized by the impossibility of the opposite. If in the instant, in which the attracted mass has moved over the width of one zone, the width of the now conterminal zone, which contains the state $s-1$, coming from the attracting mass, should contain entirely the state of zone 1, then in the time, which the attracted mass needs to move through an infinitely short distance of second order, the constrained state had to propagate over an infinitely small distance of first order, viz., it would need an infinitely high velocity, which contradicts the premiss. Furthermore, if the attracting mass stays at rest, i.e., its gravitational field does not change, and the zones $s-1$ and 1 should meet exactly, it had to be irrelevant whether one assumes that the gravitational field of the attracted mass is fixed to the mass and propagates through the gravitational field of the attracting mass with the same velocity that the former mass has, or whether one envisions that the zones of the gravitational field of the attracting mass, quasi detached from the mass, would saturate with another velocity as the latter the zones of the gravitational field of the attracting mass, which equals the premiss of a finite propagation velocity of the constrained state and another combination of the mutual zones of the mass, thus leading to a different quantity of attraction compared to the first premiss.

Consequently, the movement of the attracted mass does not only not follow Newton's law, which would be valid for the real distance of the masses at rest, but also for the deviation from it the distance of the attracted mass from the point, which the attracting mass had when its constrained state emanated from it, is irrelevant. Because the attracted mass itself also determines its own behaviour in the present position, through its velocity and the velocity of the constrained state. The situation is different from that, e.g., of the propagation of light, where the only process is that the light of one mass travels to the other one. Rather, the gravitational movement of one mass depends on the state of the space in its vicinity, and the latter is influenced by the mass itself as well as by the other masses.

However, if one tries to determine the modification of Newton's law, one finds that it is important whether the law is written in the form of the accelerating force or the potential. One can prove that it is impossible to derive the accelerating force, which the movement of the masses show, only from the assumptions of Newton's law and a finite propagation velocity of the constrained state, thus one can only calculate the potentials of the masses on each other.

Since to determine the modification, apart from the assumption of the velocity of gravitation, Newton's law is the only fundamental fact, and since the latter can only be regarded as exactly valid for the masses at rest or the transition from rest to movement, the only way to reach the aim is to show that at a given distance of the masses in motion the gravitational force has a different value from the one it has according to this law for a certain distance at rest; or more precisely, one has to look for distances for the motion, at which the constrained state at one or both masses has the same configuration as for certain other distances at rest. However, here the concept of the accelerating force fails. Since the latter is a directed quantity, dependent on the configuration of the constrained state in the vicinity of the moving mass, it makes a difference whether the constrained state is in an equilibrium, as a consequence of the masses being at rest, or whether it is constantly changing, as a consequence of the masses moving. In the former case, only the constrained state can be a directed and directing force; in the latter case, the directed and directing forces have to be added, which the masses by their movement and the constrained state by its propagation can exhibit. The analysis of the connection between these three forces would presuppose knowledge of the constrained state, which would be more than the concept that is derived from the assumption of a finite propagation velocity of gravitation and cannot be taken into account here.

Things are different when one turns towards the potential. The latter, following Newton's expression, specifies for a given distance of the attracting mass the amount of work, which has do be done when the attracted mass is moved into infinite distance, and the velocity of this movement can be neglected, thus has an arbitrary small value, going to zero. What concerns us here is the value of the work for a given movement when the velocity has a recognisable constant value. Its spatial or its spatial and temporal change multiplied by the attracted mass determines the growth or decrease of the living force and the potential energy of the gravitation between the masses during the translation from one postion to another. It also determines the accelerating forces. Therefore, the potential has also to be dependent on the constrained state of the mass it belongs to. But it is not a directed quantity, thus the directed and directing forces, which the constrained state can produce by its propagation and the mass can produce by its movement, cannot influence its value. That is to say, when the distributions of the constrained state in the vicinity of a moving and a resting mass are the same, also the potentials at the masses, which the constrained state can produce, have to be the same.

In the end, the discussed difference is that only the potential has a velocity, but not the accelerating force. Also, in my earlier thesis I commenced with the potential immediately. However, taken literally propagation can neither be attributed to the potential nor to the accelerating force. What really spreads out successively is the constrained state. But since the latter determines the potentials in the vicinity of the masses, be they at rest or in motion, one can say that the potential reaches a mass simultaneously with the constrained state. What is more, one can say that the potential reaches a place even when there is no mass; since it always has the tendency to take on the value, which follows from the constrained state, as soon as a mass is put at that place. On the other hand, since the accelerating force is determined partly by coaction of the place and the direction of motion of the constrained state with the motion and direction of the mass, it is meaningless without presence of the latter, and one cannot understand what it is supposed to mean that it propagates in space, viz., transfers from one mass to another through empty space.

But by attributing this to the potential, one does not make a final decision over the determinability of the potential's value as it really acts on the attracted mass. Since one is only calculating which value of the potential the surrounding constrained state can produce, but not has to produce under all circumstances. Again, one should picture how the attracting and the attracted masses appear suddenly in space. Furthermore, if their motion also begins suddenly then the potential appears at once. Contrarily, if one assumes that the motion begins only after a certain time, because the constrained state propagates successively, then the result would be an interruption of this continuity, if the arrival of the constrained state at the attracted mass and the attracting mass' potential reaching its full value are taken to be simultaneous. If also the extension of the attracted mass was infinitely small, then, nevertheless, the effect of the constrained state on it would not need an infinitely small time, but would be instantaneous, consequently, the effect would without time delay or in an infinitely small time of higher order extend over an infinitely small path of first order, and therefore propagate with an infinitely high velocity. Notwithstanding, the development of the potential from zero to its full value is not the previous propagation, because it happens on the mass and not in the mass-free space. The potential does merely not act at once, but successively on all parts of the mass. How long this process actually takes when the mass is at rest does not need to interest us here. In any case it is proportional to the propagation velocity of the constrained state. That is, if for some reasons the latter would become greater and the same potential, following Newton's law, would occur, then the propagation time would become smaller by the same proportion; otherwise one would have to assume that the potential of the attracting mass only shows up by the movement of the attracted mass only some time after the incoming constrained state has already passed by the attracted mass. Now, if the attracting mass has a velocity directed towards the attracted mass, then the constrained state travels from the former to the latter correspondingly more quickly, as the mass moves compared to it being at rest. Because if the constrained state travels by its own propagation through the width of one zone in a certain time, then to do this when the mass is in motion it has also to travel the distance which the mass has traveled. If, additionally to that, the attracted mass moves towards the constrained state, then both pass each other with a velocity that is the sum of their respective velocities. Consequently, if the masses are in motion, the time till the potential comes into existence is shortened by the ratio of the constrained state's propagation speed to the sum of the latter and the velocities of the masses. In that case, the potential that would be created with the masses being at rest has not enough time for its development. Accordingly, a smaller potential will be created.

If, now that the last reasoning has been presented, which is needed to prepare the deduction of the potential of one mass upon the other in motion, as I have done it in my previous thesis, then at the same time all is prepared to be able to judge older attempts to find the velocity of gravitation. It will now become clear that the concept of the velocity and its influence on the attraction of masses is more then the simple concept of velocity. Also, it has to be emphasized that the concepts, which are developed here, are only consequences of the original assumption that gravitation needs a certain time to spread out in space. The involvement of hypothetical elements in these considerations is completely avoided. Thus, what follows further from the assumption is constrained by it alone; and all methods of calculation that do not agree with it have to be insufficient.

III.

A detailed survey of older attempts to calculate the velocity of gravitation is given by Oppenheim in the annual report of the "K. K. Akademische Gymnasium" in Vienna of the academic year 1894–95. The work has the title: On the question of the propagation velocity of gravitation. Its only purpose is to report and therefore it does not incorporate any critique of the presented methods. However, the report is detailed enough to not only present the foundations of the calculations, but also the calculations themselves. Since these are not of importance here, this reference to Oppenheim's work will be sufficient.

Laplace has the merit of being the one who initiated the question of the velocity of gravitation. He did so in 1805 in the 4th volume of his "Traité de la Mécanique céleste" in the 7th chapter of the 10th book. From the beginning it was obvious to try to find the velocity in question from the orbits of the planets and the moon. Thus, one automatically looked back at the methods by which one had found the velocity of light in outer space. Laplace, however, came to realize that there is an important distinction between light and gravitation. The former cannot penetrate all bodies and can therefore be deflected, as in the eclipse of Jupiter's moons. The latter, on the other hand, when it has travelled from one celestial body to another, acts continuously and is not interrupted, even when other bodies come in between. Even nowadays it is not superfluous to bring this to attention. Zöllner's proposal from 1873, to measure the velocity of gravitation from the delay with which Hengler's horizontal pendulum shows when the sun has reached its apex, if gravitation propagates successively, has sometimes been put forward again, even though it is pointless, since from the reason given it follows that this delay does not exist. Nevertheless, one can measure the velocity of light, without it being deflected by other bodies. The method of aberration belongs there. A fixed star looks displaced in the direction in which it moves towards earth by the ratio of its velocity to the velocity of light. Laplace deemed this to be an example, to assume something similar for gravitation. What he thought was that the attraction, e.g., on a planet is not meeting the latter in a straight line from the sun, but with a forward aberration, from which it follows that it would have to move as if in addition to the attraction towards the sun there was another perturbing force, perpendicular to the radius vector, which had to be calculated proportinal to the orbital velocity of the planet and inversely proportional to the propagation velocity of gravitation. The attraction itself was incorporated by Laplace unchanged according to Newton's law.

On this the following has to be kept in mind. According to Laplace, two masses, moving only influenced by their gravitational forces, would behave as if there was no finite propagation velocity; which according to the previous considerations is wrong. And if the attraction, which occurs at a certain distance of the planet from the sun according to Newton's law, really would act at this distance, only with an aberrating direction, then this would point towards a simple action at a distance, viz., a force that acts instantaneously; which would obliterate the concept of successive propagation. Of course, not only these contradictions are where Laplace is wrong, but in that he treats the gravitational field of the sun as stationary and the motion of the planet as only dependent on the latter. But neither is the sun at rest, therefore its gravitational field is constantly renewed, and does not propagate as a whole, as it would be the case for an action at a distance with an infinite propagation velocity; nor does it alone influence the space, in which the planet moves, but it merges with the changing gravitational field of the planet. In Laplace's case this leads to a particular difficulty. Since if one incorporates the effect of the velocity of gravitation into a general law of motion, comprising Newton's law, he does not consider whether the masses also have a motion, which stems from an influence other than the attraction and, e.g., is directed perpendicular to the orbit of the moons and the planets; if, however, one does not develop such a law at the beginning, one always has to analyze, how in the spatial vicinity, in which a mass moves, the directed actions, emanating and generated by the latter and other masses and belonging to the constrained state, merge with the motion and direction of the mass itself. The first mistake of Laplace is that he neglects the action, emanating from the planet, but the even bigger mistake is that he lets the action emanating from the sun to attract the planet unchanged, whereas the real action only comes to existence when meeting the planet, moreover only would come to existence if the planet started its movement in the moment of the meeting. This mistake is understandable, since Laplace and his contemporaries were, in a manner of speaking, used to reifying the long range attraction and turn it into a tangible thing. Accordingly, they did not see that their view is incompatible with the assumption of a finite propagation velocity of gravitation.

About 40 years later another development of the concept of action at a distance in electrodynamics led to Weber's fundamental electrodynamic law. According to the origin of the law, the therein occurring new constant could not have anything to do with the successive propagation of electric attraction and repulsion. But following a proposal by Gauß, it was attempted to deduce Weber's or another fundamental law from the assumption that such a propagation exists. As a consequence, the constant gradually gained the status of the propagation velocity. Especially important in this respect are Riemann's laws from 1987 in volume 131 of the "Poggendorfsche Annalen" and the work of C. Neumann on Weber's law in his thesis on the principles of electrodynamics from 1868. As one considered how Weber's law could be an expansion of Coulomb's fundamental electric law, which has the same status as Newton's gravitational law, and could possibly be an expansion of Newton's law itself, one began to describe planetary motions using fundamental electrodynamic laws and to test whether the characteristic constant in these laws could be understood as the propagation velocity of gravitation. This was initiated by Holzmüller 1870 in volume 15 of the "Zeitschrift für Mathematik und Physik"; followed by Tisserand 1872 in volume 75 of the "Comptes rendus", Servus 1885 in his dissertation and Lévy 1890 in the 110. volume of the "Comptes rendus"; finally Oppenheim in his previously mentioned work did a calculation following Clausius' fundamental electrodynamic law.

From all these attempts nothing followed for the value of the supposedly finite propagation velocity of gravitation or for the question whether there is such a velocity. There are three reasons for this. First, it seems to have been overlooked that if by this method there should be found an agreement between the spatial transmission of gravitation and electrodynamic forces, then the constant in question would not equal the velocity of light, but $\sqrt{2}$ times the latter, because at least in Weber's law the constant has this value; consequently, by this alone Lévy's imaginary and arbitrary, since it cannot be justified, combination of Weber's and Riemann's laws, from which one obtains an agreement between the unexplained movement of Mercury's perihelion and observations using the velocity of light, is flawed. Second, the discrepancies from the velocity of light and the constant in Weber's law does not indicate whether it is really a sign of the velocity of gravitation. Third, all calculations lack certain ideas, which follow from gravitation itself, that could justify the use of one or another fundamental electrodynamic law; in general, the application will even be questionable, since the attraction of masses differs from the electric attraction and repulsion in various ways, e.g., by the double sidedness of its nature, to which it has to be added that the validity of the derived laws in their own regime is at least dubitable. Nevertheless, the efforts have not completely been in vain to apply the fundamental electrodynamic laws on motions in the sun system. They have advanced the problem, which, since Laplace, has not been discussed by anyone for decades, and they have promoted the idea that the successive propagation of gravitation, if there is such, would have to be reflected in a law of motion for the masses, which is a modification of Newton's law and at rest is identical with the latter. It is also to their credit that this important problem is now again discussed in the realm of gravitation and that there is a better, though not sufficient foundation for its solution.

This was done by Lehmann-Filhés 1885 in volume 110 of the "Astronomische Nachrichten" with the title: On the motion of planets under the assumption of a non-instantaneously propagating gravity, – and by J. von Hepperger 1888 in volume 97 of the "Sitzungsberichte der K. K. Akademie in Wien" with the title: On the propagation velocity of gravitation. The final result of Lehmann-Filhés is only negative, namely that the motion of Mercury's perihelion cannot be explained from the assumption. The aim of Hepperger's work is to calculate the smallest possible value for the velocity of gravitation that is still compatible with the astronomic facts, and has the result that it is 500 times the velocity of light. Both researchers have as a starting point the idea that an effect, which emanates from the sun and reaches a planet, left the former at a time at which the distance was different from the present one. Thus, they determine the effect following Newton's law, but only for the distance of the planet from the point the sun has when the effect emanates. Consequently, the equations of motion for the planet have their form following from Newton's law, the coordinates of the sun are different, according to the time gravitation needs to travel the distance between sun and planet and the velocity by which the sun moves in space. Of course, the equivalent is true for the action of the planet on the sun, which has the consequence that in the equations of motion the coordinates of the planet are those that it had when the action emanated from it.

This is an important advance over Laplace. While the famous author of the celestial-mechanics only took the direction, but not the value of the action, which reaches the planet, to be influenced by the successive propagation of gravitation, Lehmann-Filhés and Hepperger realize that the movement of the sun together with the time-delay due to the propagation of gravitation change the attraction acting on the planet and the motion of the planet together with the time-delay of the action on the sun relative to the displacement from the masses being at rest. Of course, any proof that the latter is insufficient to determine the motion of the sun and the planet is now superfluous, even thought Lehmann-Filhés and Hepperger think otherwise; because here, spoken with the previously used labelling, we have the case that the constrained state $s - 1$, stemming from the sun, cannot meet the state 1, stemming from the planet, but has to merge partly with state 1, stemming from itself, and partly with a just occurring state 2, and that, even if the states $s - 1$ and 1 were to meet, it would at least be dubious if they produced the same accelerating force on the planet as that which they would produce if the planet and the sun had been at rest and the motion had only just started. Nevertheless, I want to point towards these works, because the idea, embarked by Lehmann-Filhés and Hepperger, finds its necessary conclusion in them. Since if the discrepancies between the orthogonal coordinates of the sun at its current position and those of the planet at the time, when the action, now reaching the sun, emanated from it, are x_1, y_1 and z_1 and the current distance of the sun from the place the planet had at that time is r_1, then, according to Lehmann-Filhés and Hepperger, the components of the forces, acting on the sun, are proportional to ${}^{x_1}/{r_1}^3$, ${}^{y_1}/{r_1}^3$ and ${}^{z_1}/{r_1}^3$; and if one sets the discrepancy between the current orthogonal coordinates of the planet and those that the sun had, when the action reaching the planet emanated from it, to be x_2, y_2 and z_2 and calls the the current distance of the planet from the earlier position of the sun r_2, then the components of the force action on the planet are equally proportional to ${}^{x_2}/{r_2}^3$, ${}^{y_2}/{r_2}^3$ and ${}^{z_2}/{r_2}^3$. It is now easy to recognise that

$$\frac{x_1}{r_1^3} = -\frac{x_2}{r_2^3}, \quad \frac{y_1}{r_1^3} = -\frac{y_2}{r_2^3} \text{ and } \frac{z_1}{r_1^3} = -\frac{z_2}{r_2^3}$$

is not possible; because if one would assume this, then one would obtain from taking the square of and adding the three equations, provided that $x_1^2+y_1^2+z_1^2 = r_1^2$ and $x_2^2+y_2^2+z_2^2 = r_2^2$, $r_1 = r_2$ and $x_1 = x_2$, $y_1 = y_2$, $y_1 = y_2$, that the paths for the sun and the planet are equal. Consequently, according to Lehmann-Filhés and Hepperger the components of the force of the sun have values different from that of the planet, viz., there is no equality between action and reaction; this cannot be explained by some external forces acting on the sun and the planet, since the fractions ${}^{x_1}/{r_1}^3$ etc. retain their meaning, even when there are no external forces. However, a free mechanical system has to display this equality, because it is a consequence of the fact that the system's centre of mass is either at rest or moves along a straight line with constant velocity. The answer on the question why the ansatz of Lehmann-Filhés and Hepperger does not comply with the latter is the following: since the force acting on the planet depends on the position of the sun and the force acting on the sun depends on the force acting on the planet, in the first case also the motion of the planet as in the second case the motion of the sun has to have some influence. This conclusion, however, entails the requirement to consider the character and the alteration of the space enclosing the masses; because it would remain unintelligible how in an indifferent environment it could make a difference whether a force, coming out of it and reaching a mass, finds the latter in a greater or smaller movement or at rest. From here the ideas follow that culminate in the two main points, which were discussed in the second section.

The third point, which was discussed there, has neither been considered by Laplace nor by Lehmann-Filhés or Hepperger. Such a consideration also cannot be found in the completely ad hoc applications of the fundamental electrodynamic laws for the description of celestial movements. We find, however, that even when we put the latter aside the previous attempts to calculate the velocity of gravitation are completely flawed. The mechanistic theories of gravitation, beginning with Hooke and reviewed in 1897 by Drude in his presentation on action at a distance for the 69th conference of German natural scientists and physicians, and the electric theory, which was presented 1900 by H. A. Lorentz in his reflection on gravitation in the "Verhandlungen der K. Akademie der Wissenschaften" in Amsterdam, in part do not consider the propagation velocity of gravitation at all, in part do simply postulate that it is equal to the velocity of light. Thus, they have nothing to say on the question whether there is a finite propagation velocity of gravitation. Rather, the answer to this question could be of importance for them. For example, as a result of my previous thesis and the present work the theory of Lorentz is either wrong or has to be changed, since according to it the time-delay due to the propagation of gravitation does not explain the movement of Mercury's perihelion. The mechanistic theories of gravitation cannot solve the problem of the velocity of gravitation at all. They are merely pictures for the parts of gravitational processes that are not mechanical, and thus comprise unreal assumptions, with unfathomable consequences for the conclusions. If they would initiate crucial experiments and observations, then this would make them useful; however, the final result would not be due to them, but to the experiments and observations. A different judgement has to be made on theories like that of Lorentz. Their aim is to show a connection between different kinds of physical processes, which indeed is a possibility. But until now nothing like that has been found for gravitation, let alone that a result on the velocity of the latter has been gained.

IV.

I will now show, how the main points of the second section on the determination of the influence the successive propagation of gravitation on the motion of masses lead to the potential, derived in my previous work, and the consequential modification of Newton's law. As a completion, an outline of the computation of Mercury's orbit will be added.

Mass m is at point A and mass m' is at point B, both have an arbitrary small extension; the distance between them is $r - \Delta r$, where Δr is negative, since r decreases as a consequence of the attraction. The masses are initially at rest, such that the potential of m on m' becomes $\frac{\mu'}{r-\Delta r}$, where μ' is the product of m and the gravitation constant. Mass m' is fixed at point B, whereas mass m is free and in the first time element dt it covers the distance $AC = -dr$ in direction towards m'. We know: the gravitational field of m' remains unchanged, but in the meantime m sends out a new configuration of constrained states in space, and therefore the constrained state in the vicinity of m, when it reaches C, is different from the one it would have, had it stayed at one point. Thus, it is impossible to determine the potential that the constrained state wants to generate at m. However, the constrained state, which m emanates when it is at point C, reaches m' after the time $\Delta t - dt$, and has the same effect as would occur if m stayed at C. Consequently, the potential of m on m', that the constrained state is able to produce, has to be the same as would be the case for a constant distance $r - \Delta r + dr$, viz., equal to

$$\frac{\mu'}{r - \Delta r + dr} .$$

Only the respective distance between m and m' is now r, which is different, because in time Δt, m has traveled from A the distance $AD = -\Delta r$. Furthermore, the constrained state does not pass by with the velocity c, by which it normally would propagate in space, as it would be the case if m was at rest, but additionally has the velocity $-\frac{dr}{dt}$, which m has at C, resulting in a total velocity of $c - \frac{dr}{dt}$. The time until the potential reaches m' is thus shortened by the ratio of c to $c - \frac{dr}{dt}$, which has the consequence that the potential, acting on

m' and according to which the mass m' would begin to move if it would become free at that moment, is

$$\frac{\mu' c}{(r - \Delta r + dr)\left(c - \frac{dr}{dt}\right)}.$$

The differential thereof multiplied by m' gives the infinitely small increase of the living force of the masses in the subsequent time element. Thus, from the product of this potential with m' as well as from the living force one can infer the accelerating force between m and m', following Lagrange's general equations of motion. Therefore, to potential of m' on m has to be

$$\frac{\mu c}{(r - \Delta r + dr)\left(c - \frac{dr}{dt}\right)} \text{ with } \mu = \frac{\mu' m'}{m};$$

because otherwise one would come to the result that the accelerating force from m' on m does not equal the one from m on m', while one knows from reasons given above that there has to be an equality between action and reaction.

Now we assume that m was fixed at A and m' was in motion, which would result in the same considerations with the same potential, only the the distances $-dr$ and $-\Delta r$ would be BC' and BD' respectively, instead of AC and AD. Of course, in both cases it is irrelevant for how long the mass m has been at rest at A or the mass m' at B before they are released. Thus, the line of thought could immediately start with the positions D and B or A and D' of the masses and follow the same steps. The derived expressions for the potential are valid for all distances.

The only question remaining is what happens when both masses are free to move. The configuration of the constrained state, which occurs when only m' translates to D', will equally develop, but D' is a bit closer to B, since the constrained state, coming from m, approaches faster by the value of the velocity of this mass. The potential of m on m' therefore has to be determined by the same expression as follows from Newton's law for the case when both masses are at rest, as well as when m is fixed; viz., it has to follow from this law with the distance $CC' = AB - AC - BC'$, to which has to be added the influence of the velocity with which the constrained state passes by m', and which now is the sum of the three velocities of the gravitation, the mass m and the mass m'. As previously done, AB can be labelled $r - \Delta r$, if $AD + BD' = -\Delta r$; $AC + BC'$ is $-dr$; and the sum of the velocities of m and m' is $-\frac{dr}{dt}$. Consequently, for the potential of m on m' one finds

$$\frac{\mu' c}{(r - \Delta r + dr)\left(c - \frac{dr}{dt}\right)}.$$

No further proof is necessary that analogously the potential of m' on m, when both masses are in motion, is given by

$$\frac{\mu c}{(r - \Delta r + dr)\left(c - \frac{dr}{dt}\right)}$$

This potential will be labelled V. After dividing numerator and denominator by c and factoring out r one finds:

$$V = \frac{\mu}{r\left(1 - \frac{\Delta r - dr}{r}\right)\left(1 - \frac{1}{c}\frac{dr}{dt}\right)}.$$

Since in the time, in which the constrained state covers the distance $r-\Delta r+dr$, the masses cover the distance $-\Delta r+dr$, the distances have to behave analogously to the velocities, viz., it has to be

$$\frac{\Delta r - dr}{r - \Delta r + dr} = \frac{1}{c}\,\frac{dr}{dt}\,;$$

in which $\Delta r - dr$ can only be very small compared to r, because otherwise Newton's law would not be valid for moving masses. Accordingly it becomes:

$$V = \frac{\mu}{r\left(1 - \frac{1}{c}\,\frac{dr}{dt}\right)^2}\,.$$

By evaluating to second order we get:

$$V = \frac{\mu}{r}\left[1 + \frac{2}{c}\,\frac{dr}{dt} + \frac{3}{c^2}\left(\frac{dr}{dt}\right)^2\right].$$

As already was mentioned in the second section, this formula gives the work that would have to be done, if one unit of the mass m with relative velocity dr/dt had to be displaced into infinite distance, and is the variation thereof, multiplied by m, for the translation of the masses on the path dr and is the increase of the active force T in the time dt. Thus, following Lagrange's general equations of motion for the acceleration of m one gains, if $dr/dt = r'$,

$$\frac{1}{m}\,\frac{dT}{dr} - \frac{1}{m}\,\frac{d}{dt}\,\frac{dT}{dr'} = \frac{dV}{dr} - \frac{d}{dt}\,\frac{dV}{dr'} = -\frac{\mu}{r^2}\left[1 - \frac{3}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{6r}{c^2}\,\frac{d^2r}{dt^2}\right].$$

For the calculation of the motion of Mercury's perihelion the extension of the sun and the planet are irrelevant. Additionally, it is convenient to describe the orbit of the planet with the sun being the origin of the coordinate system. As a consequence, μ has to be increased by the ratio of the sum of the masses of the sun and the planet to the mass of the sun. If one sets:

$$\frac{3}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{6r}{c^2}\,\frac{d^2r}{dt^2} = F,$$

then the equations of motion of the planet are:

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3}(1-F),$$

$$\frac{d^2y}{dt^2} = -\frac{\mu y}{r^3}(1-F).$$

By multiplication of the first equation with y and the second equation with x and by subtraction we get:

$$x\,\frac{d^2y}{dt^2} - y\,\frac{d^2x}{dt^2} = 0\,.$$

From the deduction of the motion of the planet following Newton's law we know that, if θ is the angle between that radius vector and the positive axis of abscissae and L is a constant,

$$r^2\,\frac{d\vartheta}{dt} = L$$

follows. Therefore, if we set in the equations of motion:

$$dt = \frac{r^2\,d\vartheta}{L},$$

$$\frac{x}{r} = \cos\vartheta,$$

$$\frac{y}{r} = \sin\vartheta,$$

then they are:

$$d\,\frac{dx}{dt} = -\frac{\mu}{L}(1-F)\cos\vartheta\,d\vartheta\,,$$
$$d\,\frac{dy}{dt} = -\frac{\mu}{L}(1-F)\sin\vartheta\,d\vartheta\,.$$

Integration leads to, with M and N being constants:

$$\frac{dx}{dt} = -\frac{\mu}{L}\sin\vartheta + \left(M + \int\frac{\mu}{L}F\cos\vartheta\,d\vartheta\right),$$
$$\frac{dy}{dt} = \quad\frac{\mu}{L}\cos\vartheta + \left(N + \int\frac{\mu}{L}F\sin\vartheta\,d\vartheta\right).$$

So we get:

$$r = \frac{L}{\frac{\mu}{L} - \left(M + \int\frac{\mu}{L}F\cos\vartheta\,d\vartheta\right)\sin\vartheta + \left(N + \int\frac{\mu}{L}F\sin\vartheta\,d\vartheta\right)\cos\vartheta}\,.$$

This is the equation of an ellipse. If one labels its half large axis as a, the half small axis as b, the numerical eccentricity as ϵ and the angle between a and the positive axis of abscissae as ω, then one finds that, if one forms the three equations for $r = a(1-\epsilon)$, $r = a(1+\epsilon)$ and $r = \frac{b^2}{a}$,

$$L = b\sqrt{\frac{\mu}{a}}\,,$$
$$M + \int\frac{\mu}{L}F\cos\vartheta\,d\vartheta = -\frac{\varepsilon}{b}\sqrt{a\mu}\sin\omega\,,$$
$$N + \int\frac{\mu}{L}F\sin\vartheta\,d\vartheta = \quad\frac{\varepsilon}{b}\sqrt{a\mu}\cos\omega\,.$$

We now have to differentiate the last two equations with respect to θ, keeping in mind that $\frac{b}{\sqrt{a}}$ is constant; furthermore the value for L has to be inserted and the first equation has to be divided by $\frac{\sqrt{a\mu}}{b}\cos\theta$, the second by $\frac{\sqrt{a\mu}}{b}\sin\theta$. The result is:

$$F = -\frac{\sin\omega}{\cos\vartheta}\frac{d\varepsilon}{dt}\frac{dt}{d\vartheta} - \varepsilon\frac{\cos\omega}{\cos\vartheta}\frac{d\omega}{dt}\frac{dt}{d\vartheta}\,,$$
$$F = \quad\frac{\cos\omega}{\sin\vartheta}\frac{d\varepsilon}{dt}\frac{dt}{d\vartheta} - \varepsilon\frac{\sin\omega}{\sin\vartheta}\frac{d\omega}{dt}\frac{dt}{d\vartheta}\,.$$

From the equality of both equations it follows with $a = \theta - \omega$:

$$\frac{d\varepsilon}{dt} = -\varepsilon\,\mathrm{tang}\,\alpha\,\frac{d\omega}{dt}$$

and from this:

$$F = -\frac{\varepsilon}{\cos\alpha}\frac{dt}{d\vartheta}\frac{d\omega}{dt}\,.$$

But for F there can also be found another expression, in agreement with its original meaning. Applying the equations:

$$\frac{d\varepsilon}{dt} = -\varepsilon\,\mathrm{tang}\,\alpha\,\frac{d\omega}{dt}\,,$$
$$r^2\frac{d\vartheta}{dt} = L\,,$$
$$L = b\frac{\sqrt{\mu}}{\sqrt{a}}$$

one finds:

$$r = \frac{\frac{b^2}{a}}{1 + \varepsilon \cos \alpha},$$

$$\begin{aligned}
\frac{dr}{dt} &= -\frac{a r^2}{b^2}\left(\cos\alpha \frac{d\varepsilon}{dt} - \varepsilon \sin\alpha \frac{d\vartheta}{dt} + \varepsilon \sin\alpha \frac{d\omega}{dt}\right) \\
&= -\frac{a r^2}{b^2}\left(-\varepsilon \cos\alpha \operatorname{tang}\alpha \frac{d\omega}{dt} - \varepsilon \sin\alpha \frac{d\vartheta}{dt} + \varepsilon \sin\alpha \frac{d\omega}{dt}\right) \\
&= \frac{a \varepsilon r^2}{b^2} \sin\alpha \frac{d\vartheta}{dt} \\
&= \frac{\varepsilon \sqrt{a\mu}}{b} \sin\alpha,
\end{aligned}$$

$$\begin{aligned}
\frac{d^2 r}{dt^2} &= \frac{\sqrt{a\mu}}{b} \sin\alpha \frac{d\varepsilon}{dt} + \frac{\varepsilon\sqrt{a\mu}}{b} \cos\alpha \frac{d\vartheta}{dt} - \frac{\varepsilon\sqrt{a\mu}}{b} \cos\alpha \frac{d\omega}{dt} \\
&= -\frac{\varepsilon\sqrt{a\mu}}{b} \sin\alpha \operatorname{tang}\alpha \frac{d\omega}{dt} + \frac{\varepsilon\sqrt{a\mu}}{b} \cos\alpha \frac{d\vartheta}{dt} \\
&\qquad - \frac{\varepsilon\sqrt{a\mu}}{b} \cos\alpha \frac{d\omega}{dt} \\
&= -\frac{\varepsilon\sqrt{a\mu}}{b} \sin\alpha \operatorname{tang}\alpha \frac{d\omega}{dt} + \frac{\varepsilon\mu}{r^2} \cos\alpha - \frac{\varepsilon\sqrt{a\mu}}{b} \cos\alpha \frac{d\omega}{dt} \\
&= -\frac{\varepsilon\sqrt{a\mu}}{b\cos\alpha} \frac{d\omega}{dt} + \frac{\varepsilon\mu}{r^2} \cos\alpha.
\end{aligned}$$

Therefore:

$$F = \frac{3\varepsilon^2 a\mu}{b^2 c^2} \sin^2\alpha + \frac{6\varepsilon r\sqrt{a\mu}}{b c^2 \cos\alpha} \frac{d\omega}{dt} - \frac{6\varepsilon\mu}{r} \cos\alpha.$$

If one substitutes $dt/d\theta$ by $\frac{r^2\sqrt{a}}{b\sqrt{\mu}}$, one finds by equalization of the expression above with the previous expression of F that $d\omega/dt$ is now

$$\frac{\varepsilon r^2\sqrt{a}}{b\sqrt{\mu}\cos\alpha} \frac{d\omega}{dt} = -\frac{3\varepsilon^2 a\mu}{b^2 c^2} \sin^2\alpha - \frac{6\varepsilon r\sqrt{a\mu}}{b c^2 \cos\alpha} \frac{d\omega}{dt} + \frac{6\varepsilon\mu}{r} \cos\alpha,$$

which gives after introducing

$r = \frac{b^2}{a(1 + \varepsilon\cos\alpha)}$ and $b = a\sqrt{1 - \varepsilon^2}$

and divided by

$$\frac{\varepsilon r^2\sqrt{a}}{b\sqrt{\mu}\cos\alpha}$$

the expression:

$$\begin{aligned}
\frac{d\omega}{dt} &= -\frac{6\mu}{a(1-\varepsilon^2)c^2}(1 + \varepsilon\cos\alpha)\frac{d\omega}{dt} \\
&\qquad - \frac{3\varepsilon\mu^{3/2}}{a^{5/2}(1-\varepsilon^2)^{5/2}c^2}(1 + \varepsilon\cos\alpha)^2 \sin^2\alpha\cos\alpha \\
&\qquad + \frac{6\mu^{3/2}}{a^{5/2}(1-\varepsilon^2)^{5/2}c^2}(1 + \varepsilon\cos\alpha)^3\cos^2\alpha.
\end{aligned}$$

We multiply this equation with dt and set in the second and third part of the right-hand side:

$$d t = \frac{r^2}{L}\, d\vartheta = \frac{a^{3/2}(1-\varepsilon^2)^{3/2}}{\mu^{1/2}(1+\varepsilon\cos\alpha)^2}\,(d\alpha + d\omega).$$

With an appropriate arrangement and division we get:

$$= \frac{\frac{6\mu}{a(1-\varepsilon^2)c^2}(1+\varepsilon\cos\alpha)\cos^2\alpha - \frac{3\varepsilon\mu}{a(1-\varepsilon^2)c^2}\sin^2\alpha\cos\alpha}{1+\frac{6\mu}{a(1-\varepsilon^2)c^2}(1+\varepsilon\cos\alpha) - \frac{6\mu}{a(1-\varepsilon^2)c^2}(1+\varepsilon\cos\alpha)\cos^2\alpha + \frac{3\varepsilon\mu}{a(1-\varepsilon^2)c^2}\sin^2\alpha\cos\alpha}\, d\alpha.$$

If we divide the numerator and the denominator by:

$$\frac{3\mu}{a(1-\varepsilon^2)c^2} = \frac{\gamma}{c^2},$$

arrange by increasing powers of $\cos a$ and set:

$$-\varepsilon\cos\alpha + 2\cos^2\alpha + 3\varepsilon\cos^3\alpha = v\ ,$$

$$3\varepsilon\cos\alpha - 2\cos^2\alpha - 3\varepsilon\cos^3\alpha = w\ ,$$

we get:

$$d\omega = \frac{v}{\frac{c^2}{\gamma}+2+w}\, d\alpha = \left[\frac{v}{\frac{c^2}{\gamma}+2} - \frac{v w}{\left(\frac{c^2}{\gamma}+2\right)^2}\right] d\alpha .$$

This gives for the motion of the perihelion ψ during the revolution of the planet:

$$\psi = \int_0^{2\pi}\left[\frac{v}{\frac{c^2}{\gamma}+2} - \frac{v w}{\left(\frac{c^2}{\gamma}+2\right)^2}\right] d\alpha = \frac{2\pi}{\frac{c^2}{\gamma}+2} + \frac{3\pi(8-\varepsilon^2)}{8\left(\frac{c^2}{\gamma}+2\right)^2} .$$

Furthermore:

$$\frac{c^2}{\gamma} + 2 = \frac{\pi}{\psi} + \sqrt{\frac{\pi^2}{\psi^2} + \frac{3\pi(8-\varepsilon^2)}{8\psi}} .$$

Because ψ is small, the second part of the root is negligible compared to the first, thus we have:

$$\frac{c^2}{\gamma} + 2 = \frac{2\pi}{\psi},$$

$$c^2 = \frac{2\pi\gamma}{\psi} - 2\gamma .$$

Again 2γ is negligible compared to $2\pi\gamma/\psi$ and we have:

$$c^2 = \frac{6\pi\mu}{a(1-\varepsilon^2)\psi} .$$

Here we can insert, with τ being the period of revolution of the planet:

$$\begin{aligned}\mu &= \frac{4\pi^2 a^3}{\tau^2},\\ a &= 0.3871 \cdot 149 \cdot 10^6 \text{km},\\ \epsilon &= 0.2056,\\ \tau &= 88\,\text{days},\\ \psi &= 4.789 \cdot 10^{-7}.\end{aligned}$$

The result then is:

$$c = 305\,500 \frac{\text{km}}{\text{sec}}.$$

V.

It is one of the fundamental physical phenomena that the different kinds of physical processes need time for their spatial propagation. Because whether the processes act close range or at a distance and in which medium they propagate, depends on whether there is such a time consumption and what its value is. Therefore, one always endeavours for each kind of process to show the time consumption in different effects; furthermore, one will chose the most simple and straightforward examples of these effects. At the moment, in both respects, e.g., the theory of the propagation of electric waves is inferior to the theory of the propagation of light. Also for the case of gravitation my previous and the present work do not have all desirable qualities. Of course, it is impossible to investigate the spatial propagation of gravitation without the here presented modification of Newton's law. But it is possible that there are cases in which the general law has a simpler form and will therefore be easier to derive. Knowledge of the general law can help to find such cases.

I want to highlight in particular one example. The general expressions for the potential and the accelerating force for the gravitational motion, which I have derived, are valid as long as the velocity of the masses is small compared to the velocity of gravitation. It is valid as well, in cases when this velocity, being sufficiently small, is directed by external forces, under the condition that the center of mass of both masses has an approximate uniform motion. This condition is fulfilled, e.g., when a mass is uniformly lowered to earth or lifted from it. The general law then has special consequences, which can initiate suitable experiments and observations. There is only one concern remaining, namely that the expected effects are too small to be seen, because the velocity of gravitation is very high.

If this problem was to make in vain any hope to find the velocity of gravitation with different effect, then the present proof at least entitles the belief that the medium of the attraction of masses is identical to the one of light, thermal radiation, electric and magnetic attraction and repulsion, electric waves etc., and gravitation can be described as a close-range effect, without making dubious assumptions. Naturally, the theory I have in mind does not give a so called mechanistic explanation of the phenomena, but leads to facts about the connection between gravitation and other physical processes.

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[Information on the translation may be supplied by contacting Miles Osmaston (miles@osmaston.demon.co.uk).]